Gröbner bases for the power of a general linear form in an Artinian monomial complete intersection

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Problem-solving seminar in commutative algebra
Stockholm University, 26 June 2025

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Motivation

Definition

We study $I_{n,\mathbf{m},k} = (x_1^{m_1}, \dots, x_n^{m_n}, (x_1 + \cdots + x_n)^k) \subset \mathbf{k}[x_1, \dots, x_n]$, where $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 2}^n$ and $k \geq 1$.

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- Consider $P_{n,\mathbf{m}} = (x_1^{m_1}, \ldots, x_n^{m_n})$.
  The algebra
  $A_{n,\mathbf{m}} := \mathbf{k}[x_1, \ldots, x_n]/P_{n,\mathbf{m}}$
  - is Artinian.
  - is Gorenstein.
  - has the strong Lefschetz property if $\mathrm{char}(\mathbf{k}) = 0$.

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- The algebra $A_{n,\mathbf{m},k} := \mathbf{k}[x_1, \ldots, x_n]/I_{n,\mathbf{m},k}$
  - is Artinian.
  - is generated by forms.
  - has initial ideals with combinatorial structure.
  - has *thin* Hilbert series if $\mathrm{char}(\mathbf{k}) = 0$.

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Aim

Describe all reduced Gröbner bases of the ideals $I_{n,\mathbf{m},k}$ and apply them to Lefschetz properties and Hilbert series of ideals generated by general forms.

Acknowledgements

This talk is based on joint work with:

- Filip Jonsson Kling and Samuel Lundqvist (Stockholm University),
- Fatemeh Mohammadi (KU Leuven).

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Setting

- $\mathbf{k}$: field of characteristic zero.
- $R = \mathbf{k}[x_1, \ldots, x_n]$: standard graded polynomial ring.
- $\mathrm{Mon}(R)$: set of *monomials* $s = x_1^{s_1} \cdots x_n^{s_n} \in R$ with $s_i \in \mathbb{Z}_{\geq 0}$.
- *Monomial ideal*: ideal generated by monomials.
- Every monomial ideal admits a unique minimal generating set.
- $\mathrm{supp}(f)$: set of monomials in polynomial $f \in R$ with coefficient $\neq 0$.
- *Form* of degree d: $\mathbf{k}$-linear combination of monomials of degree d.

Monomial orderings and Gröbner bases

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A total ordering $\prec$ of $\mathrm{Mon}(R)$ is called *monomial ordering* if

(i) $\forall\, s \in \mathrm{Mon}(R) \setminus \{1\},\ 1 \prec s$.

(ii) $\forall\, s, t, u \in \mathrm{Mon}(R),\ s \prec t \implies u \cdot s \prec u \cdot t$.

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Let $\prec$ be a monomial ordering, and let $\{0\} \neq I \subseteq R$ be an ideal. $G = \{g_1, \dots, g_r\} \subset I \setminus \{0\}$ is a *reduced Gröbner basis* of I if...

(i) ... $\mathrm{in}(I)$ minimally generated by $\mathrm{in}(G) := \{\mathrm{in}(g_1), \dots, \mathrm{in}(g_r)\}$,

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Remark

For a fixed term order, the reduced Gröbner basis is unique.

Hilbert series

Definition

Let $M = \oplus_{d \geq 0} M_d$ be a finitely generated graded R-module with degree-d components M_d. The *Hilbert series* of M is

$$\mathrm{HS}(M; t) = \sum_{d=0}^{\infty} \dim_{\mathbf{k}}(M_d)\, t^d \,.$$

Example

For $A = \mathbf{k}[x_1, x_2]/(x_1^2, x_2^3)$, we have $\mathrm{HS}(A; t) = 1 + 2t + 2t^2 + t^3$.

Remark

Let $\{0\} \neq I \subseteq R$ be a homogeneous ideal, and $\prec$ a monomial ordering compatible with degrees. Then, $\mathrm{HS}(R/I; t) = \mathrm{HS}(R/\operatorname{in}_{\prec}(I); t)$.

Lefschetz properties (WLP and SLP)

- We consider Artinian algebras $A = R/I$ with $I \subset R$ homogeneous and 0-dimensional, so $\mathrm{HS}(A;t)$ is a polynomial.

Definition

Let $\ell \in R$ be a general linear form. A has the WLP if, for all $d \in \mathbb{N}_0$,

$$\mu_{\ell,d}\colon A_d \to A_{d+1}, \quad f \mapsto \ell \cdot f$$

are maximal rank maps (i.e., are either injective or surjective).
Similarly, A has the SLP if, for all $d, e \in \mathbb{N}_0$,

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have maximal rank.

- It suffices to exhibit *one* linear form ℓ with these properties.
- If A is defined by a monomial ideal, then it suffices to consider $\ell = x_1 + \cdots + x_n$ [Nicklasson 2018, Theorem 2.2].

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The ideals $I_{n,\mathbf{m},k}$ and Fröberg's conjecture

- Fröberg's conjecture [Fröberg 1985]: the Hilbert series for the quotient of R by m general forms of degrees $a_1, \ldots, a_m$ is

$$\left[\frac{\prod_{i=1}^{m}(1-t^{a_i})}{(1-t)^n}\right],$$

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- For $R/(x_1^{a_1}, \ldots, x_n^{a_n})$, [Stanley 1980] and [Watanabe 1989] independently proved that such rings satisfy the SLP.
- Coefficientwise lower bound for $\mathrm{HS}(R/I_{n,\mathbf{m},k}; t)$ is

$$\left[(1-t^k)\,\mathrm{HS}(R/P_{n,\mathbf{m}}; t)\right].$$

$\rightsquigarrow$ method to determine GBs using counting arguments.

Example and methods for squarefree algebra case

Write $I_{n,(2,\dots,2),k} = I_{n,k}$.

Example

The reduced Gröbner basis of $I_{5,2}$ is

$$G_{5,2} = \{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2,\\ g_{\{1,2\},5,2}, g_{\{1,3,4\},5,2}, g_{\{2,3,4\},5,2}\},$$

where

$$\begin{aligned} g_{\{1,2\},5,2} &= x_1x_2 + x_1x_3 + \cdots + x_4x_5, \\ g_{\{1,3,4\},5,2} &= x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_3x_4x_5, \\ g_{\{2,3,4\},5,2} &= x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5\,. \end{aligned}$$

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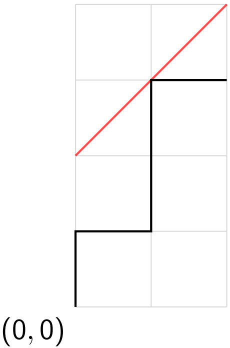
Associate lattice paths to leading monomials. $\mathrm{in}(g_{\{1,3,4\},5,2}) = x_1x_3x_4$ is shown on the right.



Results for squarefree algebra

Theorem ([JLMOS 2024])

Consider the family $\mathcal{A}$ of subsets $A \subseteq [n]$ satisfying $\max(A) = 2|A| - k$ *for some* $k \geq 2$, *and minimal with respect to inclusion.*

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$$G_{n,k} = \{x_1^2, \ldots, x_n^2\} \cup \bigcup_{d=k}^{k+\lfloor (n-k)/2 \rfloor} \{g_{A,n,k} \mid A \in \mathcal{A}, |A| = d\},$$

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where for $|A| = d$,

$$g_{A,n,k} = e_d(x_{i_1}, \ldots, x_{i_{n-d+k}})$$

is the elementary symmetric polynomial of degree d in the variables indexed by the set $\{i_1, \ldots, i_{n-d+k}\} = A \cup \{2d - k + 1, \ldots, n\}$ with leading term $x_A = \prod_{a \in A} x_a$.

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is the elementary symmetric polynomial of degree d *in the variables indexed by the set* $\{i_1, \ldots, i_{n-d+k}\} = A \cup \{2d - k + 1, \ldots, n\}$ *with leading term* $x_A = \prod_{a \in A} x_a$*. For fixed* k *the sequence* $|\{A \in \mathcal{A} : |A| = d\}|$ *is a* $(k-1)$*-fold convolution of the sequence of Catalan numbers.*

Related results

This table gives an overview of known results for the ideals $I_{n,\mathbf{m},k}$.

	$(2,\ldots,2),2$	$(2,\ldots,2),k$	$(3,\ldots,3),3$
$\mathrm{in}(\ldots)$	[BSV24]	[JLMOS24]	[BSV24]
red. GB	[JLMOS24]	[JLMOS24]	
Betti $\mathrm{in}(\ldots)$	[JLMOS24]	[JLMOS24]	
Betti $(\ldots)$	[DG+25]		
classif. WLP	[BSV24]	[JLMOS24]	[BSV24]
new proof SLP	[JLMOS24]	[JLMOS24]	

- [BSV24]: [Booth, Singh, and Vraciu, arXiv:2410.22542, 2024]
- [DG+25]: [Diethorn, Güntürkün, Hardesty, Mete, Şega, Sobieska, arXiv:2504.03977, 2025]
- [JLMOS24]: [Jonsson Kling, Lundqvist, Mohammadi, O., Sáenz-de-Cabezón, arXiv:2411.10209, 2024]

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New phenomena in non-squarefree case

- GB polynomials no longer symmetric $\rightsquigarrow$ need other description.
- More than just 0-1-Coefficients $\rightsquigarrow$ need coefficient rules.
- "Suffixes" of GB leading monomials show more variety than $x_{n-1}x_n$ $\rightsquigarrow$ need other lattice path criterion than just touching some line to describe initial ideal.

Properties of Artinian monomial complete intersections

Monomial ideals $P_{n,\mathbf{m}} = (x_1^{m_1}, \ldots, x_n^{m_n}) \in \mathbb{Z}^n_{\geq 2}$ define Artinian complete intersection algebras $A_{n,\mathbf{m}} = R/P_{n,\mathbf{m}}$ with the following properties

(i) $A_{n,\mathbf{m}}$ has the strong Lefschetz property.

(ii) The socle degree $D_{n,\mathbf{m}}$ of $A_{n,\mathbf{m}}$ is given by $D_{n,\mathbf{m}} = \sum_{i=1}^{n}(m_i - 1)$.

(iii) The Hilbert series of $A_{n,\mathbf{m}}$ is symmetric.

$$[t^d]\,\mathrm{HS}(A_{n,\mathbf{m}}; t) = [t^{D_{n,\mathbf{m}}-d}]\,\mathrm{HS}(A_{n,\mathbf{m}}; t)\,.$$

(iv) The Hilbert series of $A_{n,\mathbf{m}}$ is unimodal.

Lattice paths

Definition

A monomial $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$ is called $\mathbf{m}$-*free* if $\alpha_i < m_i$ for $1 \leq i \leq n$.

From an $\mathbf{m}$-free monomial, we obtain an $\mathbf{m}$-admissible lattice path

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- $x_i^0 \rightsquigarrow (i-1, b) \to (i, b+1) \quad$ (1-up),

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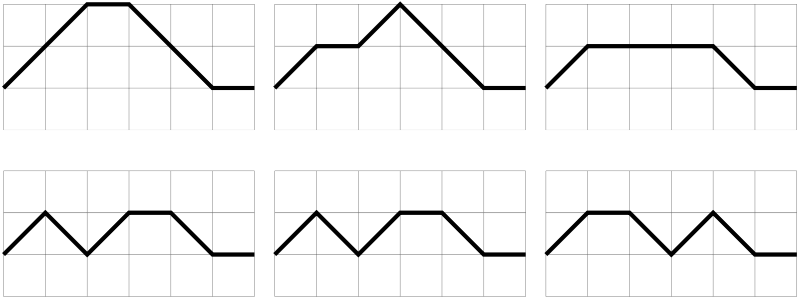
Lemma ([JLMO 2025])

Fix positive integers n, d and a vector $\mathbf{m} \in \mathbb{Z}^n_{\geq 2}$. Then the number of $\mathbf{m}$-admissible lattice paths ending at $(n, n-d)$ equals $\dim_{\mathbf{k}}(R/P_{n,\mathbf{m}})_d$.

Observation on Motzkin paths

In the case $\mathbf{m} = (3, \ldots, 3)$ and $k = 1$, associating lattice paths to GB leading monomials, observe that these paths are similar to Motzkin paths.



Number scheme and reflection points

For $1 \leq j \leq n$, let $\mathbf{m}_j = (m_1, \ldots, m_j)$. We arrange the coefficients of $\mathrm{HS}(\mathbf{k}; t),\ \mathrm{HS}(\mathbf{k}[x_1]/P_{1,\mathbf{m}_1}; t), \ldots$ as columns in a number scheme.

$$
\begin{array}{ccccc}
 & & & & 1 \\
 & & & 1 & 4 \\
 & & 1 & 3 & 8 \\
 & 1 & 2 & 4 & 10 \\
1 & 1 & 2 & 3 & 8 \\
 & 1 & 1 & 1 & 4 \\
 & & & & 1
\end{array}
$$

Figure: Here, $\mathbf{m} = (3, 2, 2, 3)$, and $k = 2$. A piecewise linear function interpolates between the points located $k/2$ below the midpoints of the columns.

Number scheme and reflection points

For $1 \leq j \leq n$, let $\mathbf{m}_j = (m_1, \ldots, m_j)$. We arrange the coefficients of $\mathrm{HS}(\mathbf{k}; t)$, $\mathrm{HS}(\mathbf{k}[x_1]/P_{1,\mathbf{m}_1}; t), \ldots$ as columns in a number scheme.

$$
\begin{array}{ccccc}
 & & & & 1 \\
 & & & 1 & 4 \\
 & & 1 & 3 & 8 \\
 & 1 & 2 & 4 & 10 \\
1 & 1 & 2 & 3 & 8 \\
 & 1 & 1 & 1 & 4 \\
 & & & & 1
\end{array}
$$

Figure: Here, $\mathbf{m} = (3, 2, 2, 3)$, and $k = 2$. A piecewise linear function interpolates between the points located $k/2$ below the midpoints of the columns.

Example

Let $\mathbf{m} = (3, 2, 2, 3)$ and $t = x_1 x_3 x_4^2$. Its path ends at the point $(4, 0) = (4, n - \deg(t))$, where it touches the red line.

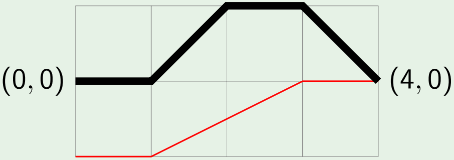


Figure: The graph of $L_{4,(3,2,2,3),2}$ and the lattice path associated with the monomial $x_1 x_3 x_4^2$.

Reflection and critical paths

Definition ([JLMO 2025])

Let $(a, b) \in \mathbb{Z}^2$ with $0 \leq a \leq n$. Reflection at the intersection of the red line and the vertical line $x = a$ yields $(a, b') =: (a, b)'$.

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$$^{i}\mathbf{P}' := (P_0, P_1, \dots, P_{i-1}, P_i', P_{i+1}', \dots, P_n')$$

is also **m**-admissible.

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$$\Lambda : \{\mathbf{P} \mid \mathbf{P} \text{ critical and } \mathbf{m}\text{-admissible}\} \to \{\mathbf{P} \mid \mathbf{P}\ \mathbf{m}\text{-admissible}\},$$
$$\mathbf{P} \mapsto {}^{\lambda(\mathbf{P})}\mathbf{P}'.$$

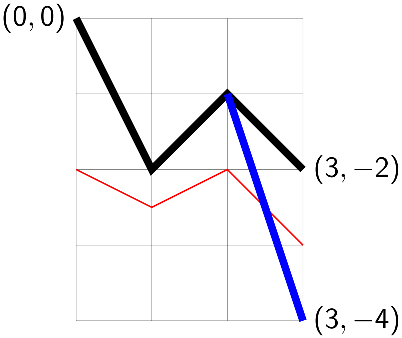
Critical path ideal

Definition ([JLMO 2025])

Let $M_{n,\mathbf{m},k}$ be the set of monomials corresponding to critical $\mathbf{m}$-admissible lattice paths. We write $(M_{n,\mathbf{m},k})$ for the ideal generated by $M_{n,\mathbf{m},k} + P_{n,\mathbf{m}}$.

Example

Consider the lattice path corresponding to the $(4, 2, 5)$-free monomial $x_1^3 x_3^2$, shown in black. Its image under the reflection map Λ is such that the initial two segments remain in black and the reflected portion appears in blue.



Degrees linked by symmetry

Remark ([JLMO 2025])

Let $d' = (\sum_{i=1}^{n} m_i) - n - d + k$. Since $R/P_{n,\mathbf{m}}$ is Gorenstein with socle degree $(d-k) + d'$, we have

$$\dim_{\mathbf{k}}(R/P_{n,\mathbf{m}})_{d-k} = \dim_{\mathbf{k}}(R/P_{n,\mathbf{m}})_{d'} \,.$$

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Furthermore, reflection of degree d path gives degree d' path, because

- A degree d admissible path ends at $Q_d = (n, n-d)$.
- A degree d' path ends at $Q_{d'} = (n, n-d') = (n, 2n + d - \sum m_i - k)$.
- The red line intersects $x = n$ at the point $Q = \left(n, \frac{3n - (\sum m_i) - k}{2}\right)$ whose y-coordinate is the average of the y-coordinates of Q_d and $Q_{d'}$

Central counting argument and Hilbert series

Definition ([JLMO 2025])

Let d be the degree of an $\mathbf{m}$-free monomial and $d' = (\sum m_i) - n - d + k$. We define the degree-wise reflection map Λ_d as the restriction of Λ to the set of critical $\mathbf{m}$-admissible paths ending at $(n, n-d)$

$$\Lambda_d : \left\{ \begin{array}{l} \mathbf{P} \text{ critical with} \\ \text{endpoint } (n, n-d) \end{array} \right\} \to \left\{ \begin{array}{l} \mathbf{P} \text{ critical with} \\ \text{endpoint } (n, n-d') \end{array} \right\}, \quad \mathbf{P} \mapsto \Lambda(\mathbf{P})\,.$$

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Proposition ([JLMO 2025])

(i) *Λ_d is invertible with $\Lambda_d^{-1} = \Lambda_{d'}$.*

(ii) *The Hilbert series of $R/(M_{n,\mathbf{m},k})$ and $R/P_{n,\mathbf{m}}$ satisfy*

$$\mathrm{HS}\left(R/(M_{n,\mathbf{m},k}); t\right) = \left[(1-t^k)\,\mathrm{HS}(R/P_{n,\mathbf{m}}; t)\right],$$

where the brackets denote truncation at the first coefficient ≥ 0.

Minimal generators (1)

Observation

Let $\mathbf{P}$ be the path of an $\mathbf{m}$-free minimal generator of $(M_{n,\mathbf{m},k})$. Then

(i) No vertex of $\mathbf{P}$ is below the red path.

(ii) The first modified edge in its reflection has maximal admissible slope.

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Definition ([JLMO 2025])

Let $s = x_1^{s_1} \cdots x_n^{s_n} \in R$, define $s_{\leq j} = x_1^{s_1} \cdots x_j^{s_j}$. For each $1 \leq j \leq n$, set

$$\overline{\mathrm{Crit}}_{n,\mathbf{m},k,j} = \left\{ s \;\middle|\; \begin{array}{l} s \text{ is } \mathbf{m}\text{-free}, x_j \text{ divides } s, \\ \deg_{x_j}(s) = 2\deg(s_{\leq j}) - k - \sum_{i=1}^{j-1} m_i + (j-1) \end{array} \right\},$$

$$\mathrm{Crit}_{n,\mathbf{m},k,j} = \left(\overline{\mathrm{Crit}}_{n,\mathbf{m},k,j} \setminus \bigcup_{\ell=1}^{j-1} \overline{\mathrm{Crit}}_{n,\mathbf{m},k,\ell} \right) \cap \mathbf{k}[x_1, \ldots, x_j].$$

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Theorem ([JLMO 2025])

The ideal $(M_{n,\mathbf{m},k})$ is minimally generated by the set

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Remark ([JLMO 2025])

The critical path ideals are nested variable-by-variable, generated by revlex segments, and their **m**-free part is strongly **m**-stable.

Table of Contents

1. Overview
2. Gröbner bases and Lefschetz properties
3. Related conjectures and results
4. Initial ideal and lattice paths
5. The reduced Gröbner basis
6. Applications and outlook

The Gröbner basis: Setup

Let $s = s' x_n^{s_n} \in R$ be an $\mathbf{m}$-free monomial satisfying the conditions

- $s_n = \deg_{x_n}(s) > 0$,
- $s_n = 2\deg(s) - k - \sum_{i=1}^{n-1} m_i + (n-1)$.

Let $R' = \mathbf{k}[x_1, \ldots, x_{n-1}, y]/(x_1^{m_1}, \ldots, x_{n-1}^{m_{n-1}})$ and $\ell = x_1 + \cdots + x_{n-1} + y$.
Let $u = x_1^{m_1-1} \cdots x_{n-1}^{m_{n-1}-1}/s'$. We define polynomials f_s and g_s in R'.

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$$g_s = \sum_{s'' | s'} \lambda_{s''} \cdot s'' \cdot y^{\deg(s) - \deg(s'')},$$

where for each monomial $s'' \mid s'$,

$$\lambda_{s''} = \left(\prod_{i=1}^{n-1} \frac{s_i!}{s_i''!} \binom{m_i - s_i'' - 1}{s_i - s_i''} \right) \cdot \frac{s_n!}{(\deg(s) - \deg(s''))!}.$$

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$$g_s = \sum_{s'' | s'} \lambda_{s''} \cdot s'' \cdot y^{\deg(s) - \deg(s'')}, \qquad f_s = \sum_{v | u} \mu_v \cdot v \cdot \ell^{\deg(s) - \deg(v) - k},$$

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$$\lambda_{s''} = \left(\prod_{i=1}^{n-1} \frac{s_i!}{s_i''!} \binom{m_i - s_i'' - 1}{s_i - s_i''} \right) \cdot \frac{s_n!}{(\deg(s) - \deg(s''))!}. \qquad \mu_v = \left(\prod_{i=1}^{n-1} \frac{s_i!}{v_i!} \binom{m_i - v_i - 1}{u_i - v_i} \right) \cdot (-1)^{\deg(v)} \cdot \frac{s_n!}{(\deg(s) - \deg(v))!}.$$

The Gröbner basis

We can show:

- $g_s = f_s \cdot \ell^k \quad$ in R'.
- Applying an obvious ring homomorphism $R' \to R/P_{n,\mathbf{m}}$, we get $\mathrm{in}(g_s) = s$ with respect to any ordering $\prec$ with $x_1 \succ \cdots \succ x_n$.
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Theorem ([JLMO 2025])

The reduced Gröbner basis of $I_{n,\mathbf{m},k}$ for $x_1 \succ \cdots \succ x_n$ is

$$G_{n,\mathbf{m},k} = \{x_1^{m_1}, \ldots, x_n^{m_n}\} \cup \{g_s \mid s \in \mathrm{Crit}_{n,\mathbf{m},k}\},$$

without $x_j^{m_j}$ if $k + \sum_{i=1}^{j-1}(m_i - 1) < m_j$. For $s = s' x_j^{\deg_{x_j}(s)} \in \mathrm{Crit}_{n,\mathbf{m},k,j}$,

$$g_s = \sum_{s'' \mid s'} \lambda_{s''} \, s''(x_j + \cdots + x_n)^{\deg(s) - \deg(s'')} \mod P_{n,\mathbf{m}}.$$

Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2025])

Let p, q, and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r|pq$. Then

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- $\operatorname{HS}(R/I_{n,\mathbf{m},k};t) = \left[(1-t^k)\operatorname{HS}(R/P_{n,\mathbf{m}};t)\right]$ follows.

Other descriptions and dependency on variable ranking

Remark ([JLMO 2025])

The GB elements can be written as $g_s = s + \sum_{t \in \mathrm{Tail}(s)} \lambda_t \cdot t$, where

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Proposition ([JLMO 2025])

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(ii) *The ideal $I_{n,\mathbf{m},k}$ has at most $n!$ distinct reduced Gröbner bases.*

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(iii) *Suppose $\mathbf{m} = (m, \ldots, m)$ with $m \geq 3$. Then $I_{m,\mathbf{m},k}$ has $(n!)/(\min\{\lfloor k/(m-1) \rfloor, n\}!)$ distinct reduced GBs.*

For the case $m = 2$ see [Jonsson Kling et al. 2024, Proposition 2.29].

Enumerative properties (1)

For $\mathbf{m} = (m_1, m_2, \ldots) \in \mathbb{Z}_{\geq 2}^{\mathbb{N}}$, we study the sequence $(g_{\mathbf{m},k}(d))_{d \geq k}$, where

$$g_{\mathbf{m},k}(d) := \left|\{s \in \mathrm{Crit}_{n,\mathbf{m},k} \mid n \gg 0,\ \deg(s) = d\}\right| .$$

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Lemma ([JLMO 2025])

For $j \geq 1$ write $R_j = \mathbf{k}[x_1, \ldots, x_j]$. Let D_j be the socle degree of $R_j/P_{j,\mathbf{m}}$ and δ_j the degree of $\mathrm{HS}(R_j/I_{j,\mathbf{m},k}; t)$. Then, for $n \geq 1$ we have:

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(iii) *For every non-negative integer*

$$e \leq \min\left\{\left(m_n - 1 - \deg_{x_n}(s)\right)/2,\ \delta_{n-1}\right\} ,$$

there is $u \in \mathrm{Crit}_{n,\mathbf{m},k}$ of degree $d + e$ with $\deg_{x_n}(u) = \deg_{x_n}(s) + 2e$. *There are $[t^{\delta_{n-1}-e}]\,\mathrm{HS}(R_{n-1}/I_{n-1,\mathbf{m},k}; t)$ such elements in $\mathrm{Crit}_{n,\mathbf{m},k}$.*

Enumerative properties (2)

Theorem ([JLMO 2025])

Let $d \geq k$ be the degree of $g_s \in G_{n,\mathbf{m},k}$ for some $s = \mathrm{in}(g_s) \in \mathrm{Crit}_{n,\mathbf{m},k}$.

(i) *The integer $n = n(d, \mathbf{m}, k)$ is uniquely determined by d, $\mathbf{m}$, and k.*

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Let $d \geq k$ be the degree of $g_s \in G_{n,\mathbf{m},k}$ for some $s = \mathrm{in}(g_s) \in \mathrm{Crit}_{n,\mathbf{m},k}$.

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(ii) *Let ϵ be the minimal degree of an element $s \in \mathrm{Crit}_{n(d,\mathbf{m},k),\mathbf{m},k}$. Set $\gamma := d - \epsilon$. Then the number of degree-d elements in the GB is*

$$g_{\mathbf{m},k}(d) = \max\left\{0,\ [t^{\delta_{n-1}-\gamma}]\,\mathrm{HS}\left(\frac{\mathbf{k}[x_1,\dots,x_{n-1}]}{P_{n-1,\mathbf{m}}};t\right) - [t^{\delta_{n-1}-\gamma-k}]\,\mathrm{HS}\left(\frac{\mathbf{k}[x_1,\dots,x_{n-1}]}{P_{n-1,\mathbf{m}}};t\right)\right\}.$$

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(iii) *Let $q \in \{1,\ldots,n\}$ be the maximal integer j with $m_j > k + D_{j-1} - 2\delta_{j-1}$. Then the maximal degree of the GB $G_{n,\mathbf{m},k}$ is*

$$k + D_{q-1} - \delta_{q-1} + \min\left\{\left\lfloor \left(m_q - 1 - (k + D_{q-1} - 2\delta_{q-1})\right)/2 \right\rfloor, \delta_{q-1}\right\}.$$

Example (1)

Example

Let $n = 5$, $\mathbf{m} = (2, 3, 2, 20, 3)$ and $k = 3$. Then the maximal integer j such that $m_j > k + D_{j-1} - 2\delta_{j-1}$ is $q = 4$. The maximal degree of an element in the Gröbner basis $G_{n,\mathbf{m},k}$ is

$$\begin{aligned} &k + D_3 - \delta_3 + \min\left\{\left\lfloor\frac{m_4 - 1 - (k + D_3 - 2\delta_3)}{2}\right\rfloor, \delta_3\right\} \\ =&3 + 4 - 3 + \min\left\{\left\lfloor\frac{20 - 1 - 1}{2}\right\rfloor, 3\right\} = 4 + 3 = 7\,. \end{aligned}$$

Indeed, Macaulay2 finds a highest degree element $x_4^7 + 7x_4^6x_5 + 21x_4^5x_5^2$.

Example (2)

Example

Let $n = 5$, $\mathbf{m} = (2, 14, 3, 11, 4)$ and $k = 11$. Then the maximal integer j such that $m_j > k + D_{j-1} - 2\delta_{j-1}$ is $q = 5$. The maximal degree of an element in the Gröbner basis $G_{n,\mathbf{m},k}$ is

$$
\begin{aligned}
&k + D_4 - \delta_4 + \min\left\{ \left\lfloor \frac{m_5 - 1 - (k + D_4 - 2\delta_4)}{2} \right\rfloor, \delta_4 \right\} \\
=&11 + 26 - 18 + \min\left\{ \left\lfloor \frac{4 - 1 - 1}{2} \right\rfloor, 18 \right\} = 19 + 1 = 20\,.
\end{aligned}
$$

Indeed, Macaulay2 finds a highest degree element $x_2^5 x_3^2 x_4^{10} x_5^3$.

Related results, again

This table gives an overview of known results for the ideals $I_{n,\mathbf{m},k}$.

	$(2,\ldots,2),2$	$(2,\ldots,2),k$	$(3,\ldots,3),3$	$\mathbf{m},k$
$\mathrm{in}(\ldots)$	[BSV24]	[JLMOS24]	[BSV24]	[JLMO25]
red. GB	[JLMOS24]	[JLMOS24]	[JLMO25]	[JLMO25]
Betti $\mathrm{in}(\ldots)$	[JLMOS24]	[JLMOS24]		
Betti $(\ldots)$	[DG+25]			
classif. WLP	[BSV24]	[JLMOS24]	[BSV24]	
new proof SLP	[JLMOS24]	[JLMOS24]	[JLMO25]	[JLMO25]

- [BSV24]: [Booth, Singh, and Vraciu, arXiv:2410.22542, 2024]
- [DG+25]: [Diethorn, Güntürkün, Hardesty, Mete, Şega, Sobieska, arXiv:2504.03977, 2025]
- [JLMOS24]: [Jonsson Kling, Lundqvist, Mohammadi, O., Sáenz-de-Cabezón, arXiv:2411.10209, 2024]

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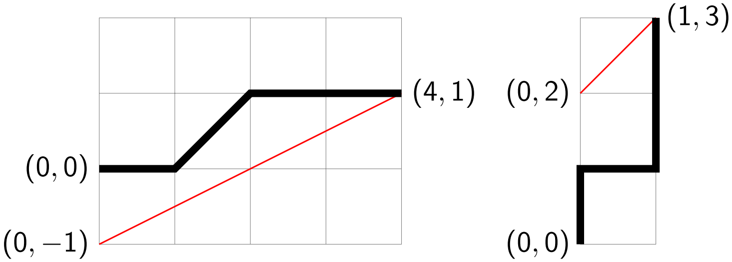
1. Overview
2. Gröbner bases and Lefschetz properties
3. Related conjectures and results
4. Initial ideal and lattice paths
5. The reduced Gröbner basis
6. Applications and outlook

Squares as special case

The lattice paths and red lines defined in the present work correspond to those of [JLMOS 2024] for $\mathbf{m} = (2, \ldots, 2)$ under the transformation

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

For $x_1 x_3 x_4 \in \mathrm{in}(I_{4,(2,2,2,2),4})$, the configurations are



This is a bijection between leading monomial paths and preserves the combinatorial information from [JLMOS24].

Cubes as special case (1)

Definition

A *Motzkin path* of length n is a lattice path from $(0,0)$ to $(n,0)$ consisting of steps that never fall below the x-axis, where each step is one of the following: an *up step* $(1,1)$, a *flat step* $(1,0)$, or a *down step* $(1,-1)$.

- The *nth Motzkin number* $M(n)$ counts the number of Motzkin paths of length n [Sloane 2018, A001006].
- The *nth Riordan number* $R(n)$ counts the number of Motzkin paths of length n with no flat steps on the x-axis [Sloane 2018, A005043].

n	0	1	2	3	4	5	6	7	8
$M(n)$	1	1	2	4	9	21	51	127	323
$R(n)$	1	0	1	1	3	6	15	36	91

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We look at the sequences $g_{(3,\ldots,3),k}$ again, which count the number of elements in the GB $G_{n,(3,\ldots,3),k}$, but shifted to start at 0.

Cubes as special case (2)

n	0	1	2	3	4	5	6	7	8
$g_{3,1}(n)$	1	0	1	1	3	6	15	36	91
$g_{3,2}(n)$	1	1	2	4	9	21	51	127	323
$g_{3,3}(n)$	1	1	3	6	15	36	91	232	603
$g_{3,4}(n)$	1	2	5	12	30	76	196	512	1353
$g_{3,5}(n)$	1	2	6	15	40	105	280	750	2025

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Theorem ([JLMO 2025])

Let $k \geq 1$, and write $k = 2q_k + r_k$ with integers $q_k \geq 0$ and $r_k \in \{0, 1\}$. Then the generating function of the sequence $g_{3,k}$ satisfies

$$\mathrm{GF}(g_{3,k}) = \big(\mathrm{GF}(M)\big)^{q_k} \cdot \big(\mathrm{GF}(R)\big)^{r_k}.$$

where M and R denote the Motzkin and Riordan number sequences.

Cubes as special case (3)

Example

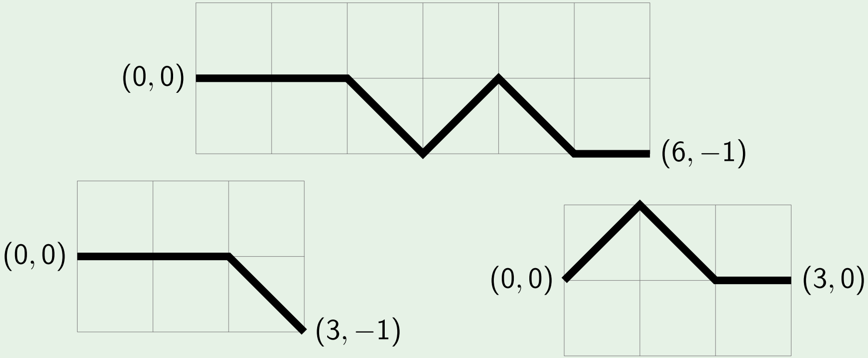

Figure: The lattice path corresponding to the Gröbner basis leading term $x_1x_2x_3^2x_5^2x_6 \in \mathrm{in}(I_{6,3,3})$, decomposed into translations of lattice paths associated with the leading terms $x_1x_2x_3^2 \in \mathrm{in}(I_{3,3,2})$ and $x_2^2x_3 \in \mathrm{in}(I_{3,3,1})$.

Quantum physics (1)

- We consider the case $\mathbf{m} = (m, \ldots, m)$ with $m \geq 2$ and $k = 1$.
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- Each step from (x, j_1) to $(x+1, j_2)$ must also satisfy

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- The states for W_0 are *decoherence-free* [Cohen et al. 2016].
- The paths from $(0,0)$ to some (N,j) (ignoring additional constraints) count the m-free terms in $\mathbf{k}[x_1, \dots, x_N]$, where $m = 2\sigma + 1$.

Quantum physics (2)

Remark

In the triangular scheme for $R/P_{i,\mathbf{m}}$, $0 \leq i \leq N$, the symmetry points of the columns now lie on the x-axis, and the red line is $y = -\frac{1}{2}$.

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Lemma ([JLMO 2025])

Let $m = 2\sigma + 1$. Consider the set of admissible lattice paths that never pass below the x-axis. These paths are in bijection with the set of $\mathbf{m}$-free monomials of degree σN in the quotient ring $\mathbf{k}[x_1, \ldots, x_N]/\operatorname{in}(I_{N,\mathbf{m},1})$.

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We define $g_{\mathbf{m},k}(z/2) = 0 \quad$ for all odd integers z.

Proposition ([JLMO 2025])

Let σ be the spin; set $m = 2\sigma + 1$ with $\mathbf{m} = (m, \ldots, m)$. Then, for any $N \in \mathbb{N}$, the degeneracy of states for W_0 is given by $g_{\mathbf{m},1}(\sigma N + 1)$.

Generalized Catalan numbers (1)

The *s-binomial coefficients* $\binom{n}{d}_s$ are defined for $n, s \in \mathbb{N}$ via

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Definition (e.g. [Ghemit and Ahmia 2024, Linz 2022])

The coefficients of the *s-Catalan triangle* are given by

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for $n > 0$ and $k \geq 0$. The s-Catalan triangle, written with entry $C^{(s)}_{n,k}$ in row n and column k, contains the s-Catalan numbers in its first column.

Log-concavity and log-convexity

- [Ghemit and Ahmia 2024] $\rightsquigarrow$
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Using our results, we provide the following lattice path interpretation of the numbers in the $(m-1)$-Catalan triangle for any positive integer $m-1$.

Proposition ([JLMO 2025])

For $m \geq 2$, $n \geq 1$, and $k \geq 0$, $C_{n,k}^{(m-1)}$ counts the m-admissible lattice paths of degree $n(m-1)-k$ whose $\mathbf{m}$-free monomials are not in $\mathrm{in}(I_{2n,\mathbf{m},1})$. The rows of the $(m-1)$-Catalan triangle are log-concave.

Outlook

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Classify the algebras $R/I_{n,\mathbf{m},k}$ with respect to the WLP.

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Open problem

Can our lattice path interpretation be used to prove log-convexity for the sequence of $(m-1)$-Catalan numbers?

Thank you for your attention!